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ON THE INTEGRABILITY OF A REPRESENTATION OF $\mathfrak{sl}(2,\mathbb{R})$

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Abstract. The Dunkl operators involve a multiplicity function k as parameter. For positive real values of this function, we consider on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ a representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ defined in terms of the Dunkl-Laplacian operator. By means of a beautiful theorem due to E. Nelson, we prove that ω_k exponentiates to a unique unitary representation Ω_k of the universal covering group $\mathfrak{G}$ of $SL(2,\mathbb{R})$. Next we show that the Dunkl transform is given by $\Omega_k(g_\circ)$, for an element $g_\circ \in \mathfrak{G}$. Finally, the representation theory is used to derive a Bochner-type identity for the Dunkl transform.

1. Introduction

In a series of papers [3, 4, 5], B. Ørsted and the present author showed that there exists an infinitesimal representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ that can be used as a crucial (and surprising) tool to treat various problems related to the theory of Dunkl operators. This representation can be thought of as analogue of the classical infinitesimal metaplectic representation of $\mathfrak{sl}(2,\mathbb{R})$. Our approach was mainly inspired by [15, 17] and [26]. See also [23, 20]. More precisely, the representation theory is used in [3] to show a Hecke-type identity for the Dunkl transform, and in [4] to prove the validity of Huygens' principle for the wave equations for the Dunkl-Laplacian operators. In [5], we prove a Harish-Chandra restriction type theorem for the Dunkl transform, i.e. the existence of an intertwining operator between the Fourier transform on Cartan motion groups and the Dunkl transform. The key to all these results is that: (i) the infinitesimal representation of $\mathfrak{sl}(2,\mathbb{R})$ exponentiates to a unitary representation of the universal covering group of $SL(2,\mathbb{R})$, and (ii) the Dunkl transform belongs to the integrated representation. The proof of the statement (ii) was given in [3] whilst the claim (i) was conjectured. In this paper, we first prove the conjecture (i) above. Our proof uses a famous result of E. Nelson [21]. Note that the integrability fact it is not obvious, since in infinite dimension, the existence of a group representation is not guaranteed from the existence of a Lie algebra representation. We should mention that the present paper deals with the "Schrödinger" model of the infinitesimal representation used in [3, 4, 5]. Next, by means of the integrated representation, we prove a Bochner-type identity for the Dunkl transform. The identity states that the Dunkl transform of a product of a radial function with a homogeneous "h-harmonic" polynomial (in the sense of Dunkl) is the product of the same "h-harmonic" polynomial with a new radial function. Further, it expresses the transform of the radial factor by means of the classical Hankel transform, which is an integral transform with the Bessel function of the first kind as kernel. The Bochner formula generalizes the Hecke-type formula for the Dunkl transform proved by Dunkl in [11], and later in [3] using a representation theory approach.

To be more precise, let $G \subset O(N)$ be a finite reflection group on $\mathbb{R}^N$ with root system $\mathscr{R}$, and choose a positive subsystem $\mathscr{R}^+$ in $\mathscr{R}$. Let $k : \mathscr{R} \to \mathbb{R}^+$, $\alpha \mapsto k_\alpha$ be a G-invariant multiplicity function, and let Δ_k be the Dunkl-Laplacian operator (see the next section for the definition). Consider the operators

$$\mathscr{J}_k := \frac{N}{2} + \gamma_k + \sum_{j=1}^{N} x_j \partial_j,$$
$$\mathbb{E} := \frac{\mathscr{J}_k - \Delta_k/2 - \|x\|^2/2}{2}, \quad \mathbb{F} := \frac{\mathscr{J}_k + \Delta_k/2 + \|x\|^2/2}{2}, \quad \mathbb{H} := \frac{\|x\|^2 - \Delta_k}{2}, \qquad (1.1)$$

where γ_k is a constant depending only on k. Due to the remarkable fact that $[\Delta_k, \|x\|^2] = 4\mathscr{J}_k$ (cf. [14, Theorem 3.3]), it follows that the elements $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{H}$ satisfy the following commutation relations

$$[\mathbb{E}, \mathbb{H}] = -2\mathbb{E}, \qquad [\mathbb{F}, \mathbb{H}] = 2\mathbb{F}, \qquad [\mathbb{E}, \mathbb{F}] = \mathbb{H}. \qquad (1.2)$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Hence, $\mathbb{E}, \mathbb{F}$ and $\mathbb{H}$ form a Lie algebra of differential operators on $\mathbb{R}^N$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. This presentation defines an action ω_k of $\mathfrak{sl}(2, \mathbb{R})$ (which is isomorphic to $\mathfrak{su}(1, 1)$) on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ by

$$\omega_k \begin{bmatrix} 1/2 & i/2 \\ i/2 & -1/2 \end{bmatrix} = \mathbb{E}, \quad \omega_k \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \mathbb{H}, \quad \omega_k \begin{bmatrix} 1/2 & -i/2 \\ -i/2 & -1/2 \end{bmatrix} = \mathbb{F}.$$

Let ϑ_k be the weight function on $\mathbb{R}^N$ defined by $\vartheta_k(x) = \prod_{\alpha \in \mathscr{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}$. For all $X \in \mathfrak{sl}(2, \mathbb{R})$, we prove that the operator $\omega_k(X)$ is skew-symmetric in the space $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$. This is a consequence of the fact that $\mathscr{J}_k^* = -\mathscr{J}_k$ in $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$ (see Proposition 3.9), and the well known fact that Δ_k is symmetric with respect to the weight ϑ_k. Moreover, we show that ω_k satisfies Nelson's criterion for a skew-symmetric representation of a Lie algebra to be integrable to a unitary representation of the corresponding simply connected Lie group [21]. Thus, ω_k exponentiates to a unique unitary representation Ω_k of the universal covering $\widetilde{SL(2, \mathbb{R})}$ of $SL(2, \mathbb{R})$, for every positive real-valued k (see Theorem 3.12). The representation Ω_k descends to $SL(2, \mathbb{R})$ if and only if $\gamma_k + \frac{N}{2} \in \mathbb{N}$, and to the metaplectic group $Mp(2, \mathbb{R})$ if and only if $\gamma_k + \frac{N}{2} \in \frac{\mathbb{N}}{2}$.

In the light of the integrability of ω_k, we show that the Dunkl transform, or the generalized Fourier transform, belongs to Ω_k. This statement was proved earlier in [3] (in view of the conjecture (i) mentioned above) using a generalized Segal-Bargmann transform associated with the Coxeter group G. However, for self-containment we give here another argument for this fact. As an application, we prove a Bochner-type identity for the Dunkl transform. The identity asserts that, if p is a homogeneous polynomial such that $\Delta_k p = 0$, then the Dunkl transform of the product of p with a radial function on $\mathbb{R}^N$ is again the product of p with a Hankel transform of the radial factor (see Theorem 3.17). This approach was used earlier by R. Howe to give a new proof for the classical Bochner identity, which corresponds to $k \equiv 0$ and $G = O(N)$ (cf. [15, 17]). We mention that the Bochner identity for the Dunkl transform was announced, without proofs, in the expository paper [5].

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2. Notations and background

Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar product in $\mathbb{R}^N$ as well as its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote by $\|x\| = \langle x, x\rangle^{1/2}$. Denote by $\mathcal{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decreasing functions equipped with the usual Fréchet space topology.

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, let r_α be the reflection in the hyperplane $\langle \alpha \rangle^\perp$ orthogonal to α

$$r_\alpha(x) := x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha, \quad x \in \mathbb{R}^N.$$

A root system is a finite spanning set $\mathscr{R} \subset \mathbb{R}^N$ of non-zero vectors such that, for every $\alpha \in \mathscr{R}$, r_α preserves $\mathscr{R}$. We shall always assume that $\mathscr{R}$ is reduced, i.e. $\mathscr{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$, for all $\alpha \in \mathscr{R}$. Each root system can be written as a disjoint union $\mathscr{R} = \mathscr{R}^+ \cup (-\mathscr{R}^+)$, where $\mathscr{R}^+$ and $(-\mathscr{R}^+)$ are separated by a hyperplane through the origin. The subgroup $G \subset O(N)$ generated by the reflections $\{r_\alpha \mid \alpha \in \mathscr{R}\}$ is called the finite reflection group associated with $\mathscr{R}$. Henceforth, we shall normalize $\mathscr{R}$ so that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in \mathscr{R}$. This simplifies formulas, without loss of generality for our purposes. We refer to [18] for more details on the theory of root systems and reflection groups.

A multiplicity function on $\mathscr{R}$ is a G-invariant function $k : \mathscr{R} \to \mathbb{C}$. Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathscr{R}$, we have $k_{h\alpha} = k_\alpha$ for all $h \in G$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathscr{R}$ is denoted by $\mathscr{K}$. If $m := \sharp\{G\text{-orbits in } \mathscr{R}\}$, then $\mathscr{K} \cong \mathbb{C}^m$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathscr{K}$, in [9], C. Dunkl defined a family of first order differential-difference operators $T_\xi(k)$ that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \langle \alpha, \xi\rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle}, \quad f \in \mathscr{C}^1(\mathbb{R}^N). \tag{2.1}$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. This definition is independent of the choice of the positive subsystem $\mathscr{R}^+$. The operators $T_\xi(k)$ are homogeneous of degree (-1). Moreover, by the G-invariance of the multiplicity function k, the Dunkl operators satisfy

$$h \circ T_\xi(k) \circ h^{-1} = T_{h\xi}(k), \qquad \forall h \in G, \tag{2.2}$$

where $h \cdot f(x) = f(h^{-1} \cdot x)$. Remarkably enough, the Dunkl operators mutually commute, i.e.

$$T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k), \qquad \forall \xi, \eta \in \mathbb{R}^N.$$

Further, if f and g are in $\mathscr{C}^1(\mathbb{R}^N)$, and at least one of them is G-invariant, then

$$T_\xi(k)[fg] = gT_\xi(k)f + fT_\xi(k)g. \tag{2.3}$$

We refer to [9, 12] for more details on the theory of Dunkl's operators.

The counterpart of the usual Laplacian is the Dunkl-Laplacian operator defined by

$$\Delta_k := \sum_{j=1}^N T_{\xi_j}(k)^2, \tag{2.4}$$

where $\{\xi_1, \dots, \xi_N\}$ is an arbitrary orthonormal basis of $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$. By the normalization $\langle \alpha, \alpha\rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle^2} \right\},$$

where Δ and ∇ denote the usual Laplacian and gradient operators, respectively (cf. [9]). It follows from (2.2) and (2.4) that Δ_k is equivariant under G,

$$h \circ \Delta_k \circ h^{-1} = \Delta_k, \qquad \forall h \in G. \tag{2.5}$$

Remark 2.1. For the j-th basis vector ξ_j, we will use the abbreviation $T_{\xi_j}(k) = T_j(k)$.

Henceforth, $\mathscr{K}^+$ denotes the set of multiplicity functions $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ such that $k_\alpha \in \mathbb{R}^+$ for all $\alpha \in \mathscr{R}$. For $k \in \mathscr{K}^+$, there exists a generalization of the usual exponential kernel $e^{\langle\cdot,\cdot\rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.2. (cf. [10, 22]) *Assume that $k \in \mathscr{K}^+$.*
(i) *There exists a unique holomorphic function E_k on $\mathbb{C}^N \times \mathbb{C}^N$ characterized by*

$$\begin{aligned} &T_\xi(k)E_k(z,w) = \langle \xi, w\rangle E_k(z,w) \qquad \forall \xi \in \mathbb{C}^N, \\ &E_k(0,w) = 1. \end{aligned} \tag{2.6}$$

Further, the Dunkl kernel E_k is symmetric in its arguments and $E_k(hz,w) = E_k(z,h^{-1}w)$ for $h \in G$ and $z, w \in \mathbb{C}^N$.
(ii) (cf. [19]) *For $x \in \mathbb{R}^N$ and $w \in \mathbb{C}^N$, we have*

$$|E_k(x,w)| \le \sqrt{|G|}e^{\|x\|\|\mathrm{Re}(w)\|}. \tag{2.7}$$

For complex-valued k, there is a detailed investigation of (2.6) by Opdam [22]. Theorem 2.2 is a weak version of Opdam's result. For integral-valued multiplicity function k, another proof for Theorem 2.2 can be found in [2] by means of a contraction procedure. When $k \equiv 0$, we have $E_0(z,w) = e^{\langle z,w\rangle}$ for $z, w \in \mathbb{C}^N$.

Let ϑ_k be the weight function on $\mathbb{R}^N$ defined by

$$\vartheta_k(x) := \prod_{\alpha\in\mathscr{R}^+} |\langle \alpha, x\rangle|^{2k_\alpha}, \qquad x \in \mathbb{R}^N.$$

It is G-invariant and homogeneous of degree $2\gamma_k$, with the index

$$\gamma_k := \sum_{\alpha\in\mathscr{R}^+} k_\alpha.$$

Let dx be the Lebesgue measure corresponding to $\langle\cdot,\cdot\rangle$, and set $L^p(\mathbb{R}^N, \vartheta_k(x)dx)$ to be the space of L^p-integrable functions on $\mathbb{R}^N$ with respect to $\vartheta_k(x)dx$. Following Dunkl [11], we define the Dunkl transform on the space $L^1(\mathbb{R}^N, \vartheta_k(x)dx)$ by

$$\mathscr{D}_k f(\xi) := c_k^{-1} \int_{\mathbb{R}^N} f(x)E_k(x, -i\xi)\vartheta_k(x)dx, \qquad \xi \in \mathbb{R}^N,$$

where c_k denotes the Mehta-type constant $c_k := \int_{\mathbb{R}^N} e^{-\|x\|^2/2}\vartheta_k(x)dx$. In view of (2.7), the transform $\mathscr{D}_k$ is well-defined. Many properties of the Euclidean Fourier transform carry over to the Dunkl transform. In particular:

Theorem 2.3. (cf. [11, 19]) *If $k \in \mathscr{K}^+$, then:*
(i) *The Dunkl transform is a homeomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. Its inverse is given by $\mathscr{D}_k^{-1}f(\xi) = \mathscr{D}_k f(-\xi)$.*
(ii) *If $f \in L^1(\mathbb{R}^N, \vartheta_k(x)dx) \cap L^2(\mathbb{R}^N, \vartheta_k(x)dx)$, then $\mathscr{D}_k(f) \in L^2(\mathbb{R}^N, \vartheta_k(x)dx)$ and $\|\mathscr{D}_k(f)\|_2 = \|f\|_2$. Further, $\mathscr{D}_k$ extends uniquely from $L^1(\mathbb{R}^N, \vartheta_k(x)dx)\cap L^2(\mathbb{R}^N, \vartheta_k(x)dx)$ to a unitary operator on $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$.*

To conclude this section, we mention that the Dunkl operators are anti-symmetric with respect to the weight function ϑ_k (cf. [9]): if $f \in \mathcal{S}(\mathbb{R}^N)$ and g is smooth such that both g and $T_\xi(k)g$ are at most of polynomial growth, then

$$\int_{\mathbb{R}^N} (T_\xi(k)f)(x)g(x)\vartheta_k(x)dx = -\int_{\mathbb{R}^N} f(x)(T_\xi(k)g)(x)\vartheta_k(x)dx. \tag{2.8}$$

3. A unitary representation of $\widetilde{SL(2,\mathbb{R})}$ and a Bochner identity

Choose $x_1, \ldots, x_N$ as the usual system of coordinates on $\mathbb{R}^N$, and recall the $\mathfrak{sl}(2,\mathbb{R})$-triple $\{\mathbb{E}, \mathbb{F}, \mathbb{H}\}$ from the introduction (see (1.1) and (1.2)). What makes $\{\mathbb{E}, \mathbb{F}, \mathbb{H}\}$ important is the fact that $\mathbb{H}$ is the infinitesimal generator of the maximal compact subgroup $SO(2,\mathbb{R})$ of $SL(2,\mathbb{R})$. Recall that the commutation relations (1.2) are due to the remarkable fact that

$$[\Delta_k, \|x\|^2] = 4\mathscr{I}_k, \tag{3.1}$$

(cf. [14, Theorem 3.3]).

Define the following three spaces

$$\mathfrak{sl}_2^+ := \mathrm{Span}\{\mathbb{E}\}, \quad \mathfrak{sl}_2^0 := \mathrm{Span}\{\mathbb{H}\}, \quad \mathfrak{sl}_2^- := \mathrm{Span}\{\mathbb{F}\}.$$

The direct sum

$$\mathfrak{g} := \mathfrak{sl}_2^+ \oplus \mathfrak{sl}_2^0 \oplus \mathfrak{sl}_2^- \tag{3.2}$$

is preserved under the usual operator bracket and is isomorphic, as a Lie algebra, to $\mathfrak{sl}(2,\mathbb{R})$. This presentation defines an action ω_k of $\mathfrak{sl}(2,\mathbb{R})$ (which is isomorphic to $\mathfrak{su}(1,1)$) on $\mathcal{S}(\mathbb{R}^N)$. The decomposition (3.2) is an instance of the Cartan decomposition

$$\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-,$$

with

$$\omega_k \begin{bmatrix} 1/2 & i/2 \\ i/2 & -1/2 \end{bmatrix} = \frac{\mathscr{I}_k - \Delta_k/2 - \|x\|^2/2}{2}, \tag{3.3}$$

$$\omega_k \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{\|x\|^2 - \Delta_k}{2}, \tag{3.4}$$

$$\omega_k \begin{bmatrix} 1/2 & -i/2 \\ -i/2 & -1/2 \end{bmatrix} = \frac{\mathscr{I}_k + \Delta_k/2 + \|x\|^2/2}{2}. \tag{3.5}$$

Henceforth, we will write

$$\mathbf{k} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathbf{n}^+ = \begin{bmatrix} 1/2 & i/2 \\ i/2 & -1/2 \end{bmatrix}, \quad \mathbf{n}^- = \begin{bmatrix} 1/2 & -i/2 \\ -i/2 & -1/2 \end{bmatrix}.$$

On the other hand, for $h \in G$, denote by $\pi(h)$ the "left regular" action of the Coxeter group G on $\mathcal{S}(\mathbb{R}^N)$

$$\pi(h)f(x) = f(h^{-1}x).$$

The actions of G and $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R}^N)$ commute. This is a consequence of (2.5) and the fact that $\mathscr{I}_k = (1/4)(\Delta_k \circ \|x\|^2 - \|x\|^2\Delta_k)$ (see (3.1)).

To investigate the structure of the representation ω_k, we first note that for a polynomial $p \in \mathcal{P}(\mathbb{R}^N)$, we have

$$e^{\nu\|x\|^2/2} p(-T(k)) e^{-\nu\|x\|^2/2} = p(\nu x - T(k)) \qquad \text{for all } \nu \in \mathbb{R}. \tag{3.6}$$

Here $p(T(k))$ is the operator derived from $p(x)$ by replacing x_j by $T_j(k)$ (recall Remark 2.1 for the notation). The identity (3.6) follows from the product rule (2.3). In particular, if $p(x) = \sum_{j=1}^N x_j^2$, equation (3.6) becomes

$$e^{\nu\|x\|^2/2}\Delta_k e^{-\nu\|x\|^2/2} = \nu^2\|x\|^2 + \Delta_k - \nu\sum_j \Big(x_jT_j(k) + T_j(k)x_j\Big). \tag{3.7}$$

Further, using the definition (2.4) of Δ_k, it follows that

$$[\Delta_k, \|x\|^2] = 2\sum_j \Big(x_jT_j(k) + T_j(k)x_j\Big).$$

Thus, in view of (3.1), equation (3.7) becomes

$$e^{\nu\|x\|^2/2}\Delta_k e^{-\nu\|x\|^2/2} = \nu^2\|x\|^2 + \Delta_k - 2\nu\mathscr{J}_k, \qquad \text{for all } \nu \in \mathbb{R}.$$

This formula allowes us to rewrite the $\mathfrak{sl}(2)$-triple $\{\mathbb{E}, \mathbb{F}, \mathbb{H}\}$ as

$$\mathbb{E} = -\frac{1}{4}e^{\|x\|^2/2}\Delta_k e^{-\|x\|^2/2}, \tag{3.8}$$

$$\mathbb{F} = \frac{1}{4}e^{-\|x\|^2/2}\Delta_k e^{\|x\|^2/2}, \tag{3.9}$$

$$\mathbb{H} = e^{-\|x\|^2/2}\Big(\mathscr{J}_k - \frac{\Delta_k}{2}\Big)e^{\|x\|^2/2}. \tag{3.10}$$

According to (3.9), the kernel of $\mathbb{F}$ consists of functions of the form $e^{-\|x\|^2/2}P(x)$, where P is h-harmonic, i.e. $\Delta_k P = 0$. Now by (3.10), we get $\mathbb{H}(e^{-\|x\|^2/2}P(x)) = e^{-\|x\|^2/2}\mathscr{J}_k P(x)$. Thus, $e^{-\|x\|^2/2}P(x)$ is an eigenvector for $\mathbb{H}$ with eigenvalue $(|\mathbf{m}| + \frac{N}{2} + \gamma_k)$ if and only if P is a homogeneous polynomial of degree $|\mathbf{m}|$. Here $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N$ and $|\mathbf{m}| = m_1 + \cdots + m_N$. In conclusion, P is a h-harmonic homogeneous polynomial of degree $|\mathbf{m}|$ and

$$\mathbb{H}\big(e^{-\|x\|^2/2}P(x)\big) = \Big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\Big)e^{-\|x\|^2/2}P(x).$$

Henceforth, for $\mathbf{m} \in \mathbb{Z}_+^N$, we set $\mathcal{H}_{|\mathbf{m}|,k}$ to be the space of h-harmonic homogeneous polynomials on $\mathbb{R}^N$ of degree $|\mathbf{m}|$.

On the other hand, for $s \in \mathbb{N}$ and $P \in \mathcal{H}_{|\mathbf{m}|,k}$, the vectors $v_s := \mathbb{E}^s\big(e^{-\|x\|^2/2}P(x)\big)$ are eigenvectors for $\mathbb{H}$ with eigenvalues $N/2 + \gamma_k + |\mathbf{m}| + 2s$. Further, the vectors v_s form an orthonormal basis for the space of the representation. Denote by $\mathcal{W}_{N/2+\gamma_k+|\mathbf{m}|}$ the $\mathfrak{sl}(2,\mathbb{R})$-representation with lowest weight $N/2 + \gamma_k + |\mathbf{m}|$. Moreover, for every ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^+)$ and $P \in \mathcal{H}_{|\mathbf{m}|,k}$, one can check that

$$\mathscr{J}_k\Big(P(x)\psi(\|x\|^2)\Big) = \Big\{(|\mathbf{m}| + N/2 + \gamma_k)\psi(\|x\|^2) + 2\|x\|^2\psi'(\|x\|^2)\Big\}P(x), \tag{3.11}$$

and

$$\Delta_k\Big(P(x)\psi(\|x\|^2)\Big) = 4\Big\{\|x\|^2\psi''(\|x\|^2) + (|\mathbf{m}| + N/2 + \gamma_k)\psi'(\|x\|^2)\Big\}P(x). \tag{3.12}$$

To prove (3.12) one needs to use the identity (3.1), i.e. $\sum_{j=1}^N (x_jT_j(k) + T_j(k)x_j) = 2\mathscr{J}_k$. Thus, for every $s \in \mathbb{N}$ and $P \in \mathcal{H}_{|\mathbf{m}|,k}$, the operator $\mathbb{E}^s$ leaves the set $\mathcal{I}P := \{\psi(\|\cdot\|^2)P \mid \psi \in \mathcal{S}(\mathbb{R}^+)\}$ invariant. In particular, the vectors v_s belong to the space

$e^{-\|x\|^2/2}\mathcal{P}(\mathbb{R}^N)$, which is dense in $\mathcal{S}(\mathbb{R}^N)$. We summarize the consequences of the above computations.

Theorem 3.1. (cf. [4]) *Assume that $k \in \mathscr{K}^+$ and $N \geq 1$. Let $\mathfrak{k} = \mathfrak{so}(2)$ be the Lie algebra of the compact group $SO(2,\mathbb{R})$.*

(i) *The space $\sum_{\mathbf{m}\in\mathbb{Z}_+^N} \mathcal{H}_{|\mathbf{m}|,k}\cdot\mathcal{I}(\mathbb{R}^N)$, where $\mathcal{I}(\mathbb{R}^N)$ denotes the space of $O(N)$-invariant Schwartz functions on $\mathbb{R}^N$, is dense in $\mathcal{S}(\mathbb{R}^N)$.*

(ii) *As a $G\times\mathfrak{sl}(2,\mathbb{R})$-module, the $G\times\mathfrak{k}$-finite vectors in the Schwartz space admit the following decomposition*

$$\mathcal{S}(\mathbb{R}^N)_{G\times\mathfrak{k}} = \bigoplus_{m\in\mathbb{Z}^+} \widetilde{\mathcal{H}}_{|\mathbf{m}|,k} \otimes \mathcal{W}_{|\mathbf{m}|+\frac{N}{2}+\gamma_k},$$

where $\mathcal{W}_{|\mathbf{m}|+\frac{N}{2}+\gamma_k}$ is the $\mathfrak{sl}(2,\mathbb{R})$-representation of lowest weight $|\mathbf{m}|+\frac{N}{2}+\gamma_k$, and $\widetilde{\mathcal{H}}_{|\mathbf{m}|,k}$ is the $O(N)$-irreducible module $e^{-\|x\|^2/2}\mathcal{H}_{|\mathbf{m}|,k}$. The summands are mutually orthogonal with respect to the inner product on $L^2(\mathbb{R}^N,\vartheta_k(x)dx)$.

Remark 3.2. (i) The decomposition in (ii) could just as well be formulated for the space $L^2(\mathbb{R}^N,\vartheta_k(x)dx)$ as for the Schwartz space.

(ii) Those readers who are familiar with the theory of Howe reductive dual pairs [16] will find that our formulation can be thought of as analogue of this theory.

The following is then immediate.

Corollary 3.3. *Under the action of $\mathfrak{sl}(2,\mathbb{R})$, the $\mathfrak{k}$-finite vectors in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ decompose as*

$$\mathcal{S}(\mathbb{R}^N)_{\mathfrak{k}} = \bigoplus_{\mathbf{m}\in\mathbb{Z}_+^N} \dim(\widetilde{\mathcal{H}}_{|\mathbf{m}|,k})\mathcal{W}_{|\mathbf{m}|+\frac{N}{2}+\gamma_k},$$

where $\dim(\widetilde{\mathcal{H}}_{|\mathbf{m}|,k}) = \begin{pmatrix} |\mathbf{m}|+N-1 \\ N-1 \end{pmatrix} - \begin{pmatrix} |\mathbf{m}|+N-3 \\ N-1 \end{pmatrix}$. If $N>1$, $\dim(\widetilde{\mathcal{H}}_{|\mathbf{m}|,k})$ is always nonzero, but if $N=1$, it is zero for $|\mathbf{m}|\geq 2$.

Remark 3.4. Above we used the fact that the map $\Delta_k : \mathcal{P}_{|\mathbf{m}|} \longrightarrow \mathcal{P}_{|\mathbf{m}|-2}$ is surjective. Thus, $\dim(\mathcal{H}_{|\mathbf{m}|,k}) = \dim(\ker(\Delta_k)_{|\mathcal{P}_{|\mathbf{m}|}}) = \dim(\mathcal{P}_{|\mathbf{m}|}) - \dim(\mathcal{P}_{|\mathbf{m}|-2})$. We refer to [12] for more details on the space $\mathcal{H}_{|\mathbf{m}|,k}$.

Theorem 3.1(i) points us to consider the map

$$\alpha^N_{\mathbf{m},k} : \mathcal{H}_{|\mathbf{m}|,k}\otimes\mathcal{S}(\mathbb{R}^+) \to \mathcal{S}(\mathbb{R}^N)$$

defined by

$$\alpha^N_{\mathbf{m},k}(h\otimes\psi)(x) := h(x)\psi(\|x\|^2) \tag{3.13}$$

with $h\in\mathcal{H}_{|\mathbf{m}|,k}$ and $\psi\in\mathcal{S}(\mathbb{R}^+)$. Moreover, using the representation ω_k, we construct a representation $\pi^N_{\mathbf{m},k}$ of $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R}^+)$ by

$$\alpha^N_{\mathbf{m},k}(h\otimes\pi^N_{\mathbf{m},k}(X)\psi) = \omega_k(X)(\alpha^N_{\mathbf{m},k}(h\otimes\psi)), \qquad X\in\mathfrak{sl}(2,\mathbb{R}). \tag{3.14}$$

Write $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{g}_{\mathbb{C}}=\mathfrak{sl}(2,\mathbb{C})$, and let $\{\mathbf{h},\mathbf{e}^+,\mathbf{e}^-\}\subset\mathfrak{g}$ be the $\mathfrak{sl}(2,\mathbb{R})$-triple given by

$$\mathbf{h} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{e}^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{e}^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

One checks easily that

$$\mathbf{h} = \mathbf{n}^+ + \mathbf{n}^-, \quad \mathbf{e}^+ = \frac{i}{2}(\mathbf{k} - (\mathbf{n}^+ - \mathbf{n}^-)), \quad \mathbf{e}^- = -\frac{i}{2}(\mathbf{k} + (\mathbf{n}^+ - \mathbf{n}^-)). \tag{3.15}$$

The equations (3.11) and (3.12) imply that the action of the basis $\{\mathbf{h}, \mathbf{e}^+, \mathbf{e}^-\}$ on $\mathcal{S}(\mathbb{R}^+)$ is given by the formulas:

$$\pi^N_{\mathbf{m},k}(\mathbf{h}) = 2t\frac{\mathrm{d}}{\mathrm{d}t} + \big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big), \tag{3.16}$$

$$\pi^N_{\mathbf{m},k}(\mathbf{e}^+) = i\frac{t}{2}, \tag{3.17}$$

$$\pi^N_{\mathbf{m},k}(\mathbf{e}^-) = 2it\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2i\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big)\frac{\mathrm{d}}{\mathrm{d}t}, \tag{3.18}$$

where t denotes the positive variable of $\mathbb{R}^+$. The following is then immediate.

Lemma 3.5. *The action of* $\mathbf{k}^+, \mathbf{n}^+$, *and* $\mathbf{n}^-$ *on* $\mathcal{S}(\mathbb{R}^+)$ *is given by*

$$\begin{aligned}
\pi^N_{\mathbf{m},k}(\mathbf{k}) &= -2t\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big)\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{2},\\
\pi^N_{\mathbf{m},k}(\mathbf{n}^+) &= -t\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \Big(\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big) - t\Big)\frac{\mathrm{d}}{\mathrm{d}t} - \frac{t}{4} + \frac{1}{2}\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big),\\
\pi^N_{\mathbf{m},k}(\mathbf{n}^-) &= t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \Big(\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big) + t\Big)\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{4} + \frac{1}{2}\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big).
\end{aligned}$$

Remark 3.6. (i) Observe that $\pi^N_{\mathbf{m},k}$ depends only on $|\mathbf{m}| + \frac{N}{2} + \gamma_k$.

(ii) The infinitesimal representation $\pi^N_{\mathbf{m},k}$ appears also in [7] and in [20] (denoted by λ_α and π_r, respectively) from a different point of view and for a different reason.

Let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, and let $\mathscr{C}$ be the quadratic Casimir element corresponding to the Killing form of $\mathfrak{g}_\mathbb{C}$. By [17, Chap. I, Eq. (1.3.8)], we have

$$\mathscr{C} = \mathbf{h}^2 + 2\mathbf{e}^+\mathbf{e}^- + 2\mathbf{e}^-\mathbf{e}^+. \tag{3.19}$$

Proposition 3.7. *The differential operator* $\pi^N_{\mathbf{m},k}(\mathscr{C})$ *is the scalar operator given by*

$$\pi^N_{\mathbf{m},k}(\mathscr{C}) = \big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big)\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k - 2\big). \tag{3.20}$$

Proof. Clearly we have

$$\pi^N_{\mathbf{m},k}(\mathbf{h}^2) = 4t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 4(|\mathbf{m}| + \frac{N}{2} + \gamma_k + 1)t\frac{\mathrm{d}}{\mathrm{d}t} + (|\mathbf{m}| + \frac{N}{2} + \gamma_k)^2, \tag{3.21}$$

and

$$\pi^N_{\mathbf{m},k}(\mathbf{e}^+\mathbf{e}^- + \mathbf{e}^-\mathbf{e}^+) = -2t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2(|\mathbf{m}| + \frac{N}{2} + \gamma_k)t\frac{\mathrm{d}}{\mathrm{d}t} - (|\mathbf{m}| + \frac{N}{2} + \gamma_k). \tag{3.22}$$

Now (3.20) follows from (3.21), (3.22) and (3.19). □

Equation (3.14) and Proposition 3.7 now combine to give the following:

Corollary 3.8. *For all* $h \in \mathcal{H}_{|\mathbf{m}|,k}$ *and* $\psi \in \mathcal{S}(\mathbb{R}^+)$, *we have*

$$\omega_k(\mathscr{C})\alpha^N_{\mathbf{m},k}(h \otimes \psi) = \big(|\mathbf{m}| + \frac{N}{2} + \gamma_k\big)\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k - 2\big)\alpha^N_{\mathbf{m},k}(h \otimes \psi).$$

Recall that $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$ denotes the space of square integrable functions on $\mathbb{R}^N$ with respect to the weighted measure $\vartheta_k(x)dx$. If f and g belong to $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$, we will write

$$\langle\langle f, g\rangle\rangle_k := \int_{\mathbb{R}^N} f(x)\overline{g(x)}\vartheta_k(x)dx.$$

Proposition 3.9. *For every $X \in \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, the operator $\omega_k(X)$ is skew-symmetric.*

Proof. Let us define the conjugate linear map on $\mathfrak{g}_{\mathbb{C}}$ by $X \mapsto X^*$, so that $X^* = -X$ for $X \in \mathfrak{g}$. We shall prove that

$$\langle\langle \omega_k(X)f, g\rangle\rangle_k = \langle\langle f, \omega_k(X^*)g\rangle\rangle_k, \tag{3.23}$$

for every $X \in \mathfrak{g}_{\mathbb{C}}$. However, since $X \mapsto \langle\langle f, \omega_k(X^*)g\rangle\rangle_k$ is complex linear, it suffices to prove (3.23) only for the basis $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$, for instance. We first note that

$$\mathbf{k} = i(\mathbf{e}^- - \mathbf{e}^+), \quad \mathbf{n}^+ = \frac{1}{2}(\mathbf{h} + i(\mathbf{e}^+ + \mathbf{e}^-)), \quad \mathbf{n}^- = \frac{1}{2}(\mathbf{h} - i(\mathbf{e}^+ + \mathbf{e}^-)).$$

Thus $\mathbf{k}^* = \mathbf{k}$ and $(\mathbf{n}^+)^* = -\mathbf{n}^-$. Now the anti-symmetry of the Dunkl operators with respect to $\vartheta_k(x)dx$ (see (2.8)), implies the symmetry of Δ_k with respect to the same weighted measure, and therefore (3.23) holds for $X = \mathbf{k}$. On the other hand, since $(\mathbf{n}^+)^* = -\mathbf{n}^-$, it suffices then only to prove that

$$\langle\langle \omega_k(\mathbf{n}^+)f, g\rangle\rangle_k = -\langle\langle f, \omega_k(\mathbf{n}^-)g\rangle\rangle_k.$$

By (1.1), this is equivalent to proving

$$\langle\langle \mathscr{I}_k f, g\rangle\rangle_k = -\langle\langle f, \mathscr{I}_k g\rangle\rangle_k.$$

One can see this from the following:

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\mathscr{I}_k f)(x)g(x)\vartheta_k(x)dx \\
&\quad= \int_{\mathbb{R}^N} \big\{\sum_j x_j\partial_j f(x)\big\}g(x)\vartheta_k(x)dx + \Big(\gamma_k + \frac{N}{2}\Big)\int_{\mathbb{R}^N} f(x)g(x)\vartheta_k(x)dx \\
&\quad= -\int_{\mathbb{R}^N} f(x)\big\{\sum_j \partial_j x_j g(x)\big\}\vartheta_k(x)dx - \int_{\mathbb{R}^N} f(x)g(x)\big\{\sum_j x_j\partial_j\vartheta_k(x)\big\}dx \\
&\qquad+ \Big(\gamma_k + \frac{N}{2}\Big)\int_{\mathbb{R}^N} f(x)g(x)\vartheta_k(x)dx \\
&\quad= -(N + 2\gamma_k)\int_{\mathbb{R}^N} f(x)g(x)\vartheta_k(x)dx - \int_{\mathbb{R}^N} f(x)\big\{\sum_j x_j\partial_j g(x)\big\}\vartheta_k(x)dx \\
&\qquad+ \Big(\gamma_k + \frac{N}{2}\Big)\int_{\mathbb{R}^N} f(x)g(x)\vartheta_k(x)dx \\
&\quad= -\int_{\mathbb{R}^N} f(x)(\mathscr{I}_k g)(x)\vartheta_k(x)dx.
\end{aligned}$$

Above we used the fact that $\sum_{j=1}^N x_j\partial_j\vartheta_k(x) = 2\gamma_k\vartheta_k(x)$, since ϑ_k is homogenous of degree $2\gamma_k$. $\square$

Recall that $\mathcal{H}_{|\mathbf{m}|,k} = \mathcal{P}_{|\mathbf{m}|} \cap (\mathrm{Ker}\Delta_k)$ denotes the space of h-harmonic homogenous polynomials of degree $|\mathbf{m}| = m_1 + \cdots + m_N$. Let $d\omega$ be the normalized rotation-invariant measure on the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. It is well known that $L^2(\mathbb{S}^{N-1}, \vartheta_k(\theta)d\omega(\theta)) = \sum^{\oplus}_{\mathbf{m}\in\mathbb{Z}_+^N} \mathcal{H}_{|\mathbf{m}|,k}$. Let $\{h_j^{(\mathbf{m})}\}_{j\in J_{|\mathbf{m}|}}$ be an orthonormal basis of $\mathcal{H}_{|\mathbf{m}|,k}$. Further, for $\mathbf{m} \in \mathbb{Z}_+^N$, $j \in J_{|\mathbf{m}|}$, and a non-negative integer ℓ, define

$$c_{\ell,\mathbf{m}} := \left(\frac{\Gamma(N/2)\ell!}{\pi^{N/2}\Gamma((N/2)+\gamma_k+|\mathbf{m}|+\ell)} \right)^{1/2},$$

and

$$\phi_{\ell,\mathbf{m},j}(x) := c_{\ell,\mathbf{m}} h_j^{(\mathbf{m})}(x) L_\ell^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(\|x\|^2) e^{-\|x\|^2/2}.$$

Here L_ℓ^α denotes the classical Laguerre polynomial given by

$$L_\ell^\alpha(x) = \frac{(\alpha+1)_\ell}{\ell!} \sum_{j=0}^{\ell} \frac{(-\ell)_j}{(\alpha+1)_j} \frac{x^j}{j!}.$$

By [11, Proposition 2.4, Theorem 2.5], the functions

$$\phi_{\ell,\mathbf{m},j}, \qquad \ell \in \mathbb{N},\ \mathbf{m} \in \mathbb{Z}_+^N,\ j \in J_{|\mathbf{m}|}$$

form an orthonormal basis of $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$.

Proposition 3.10. *The dense subspace, in $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$, spanned by the functions $\{\phi_{\ell,\mathbf{m},j} \mid \ell \in \mathbb{N}, \mathbf{m} \in \mathbb{Z}_+^N, j \in J_{|\mathbf{m}|}\}$, is stable under the action of $\omega_k(\mathfrak{sl}(2,\mathbb{C}))$. More precisely*

$$\begin{aligned}
\omega_k(\mathbf{k})\phi_{\ell,\mathbf{m},j}(x) &= \big(|\mathbf{m}| + \frac{N}{2} + \gamma_k + 2\ell\big)\phi_{\ell,\mathbf{m},j}(x), \\
\omega_k(\mathbf{n}^+)\phi_{\ell,\mathbf{m},j}(x) &= (\ell+1)\phi_{\ell+1,\mathbf{m},j}(x), \\
\omega_k(\mathbf{n}^-)\phi_{\ell,\mathbf{m},j}(x) &= -\big(|\mathbf{m}| + N/2 + \gamma_k + \ell - 1\big)\phi_{\ell-1,\mathbf{m},j}(x),
\end{aligned}$$

with $\phi_{-1,\mathbf{m},j} \equiv 0$. We may think of $\omega_k(\mathbf{n}^+)$ and $\omega_k(\mathbf{n}^-)$ as a creation and an annihilation operators, respectively.

Proof. Using the following well known recursion relations (cf. [27, Section 6.14])

$$\begin{aligned}
&t\,\mathrm{d}^2/\mathrm{d}t^2\, L_\ell^\alpha(t) + (\alpha+1-t)\mathrm{d}/\mathrm{d}t\, L_\ell^\alpha(t) = -\ell L_\ell^\alpha(t), \\
&t\,\mathrm{d}/\mathrm{d}t\, L_\ell^\alpha(t) = \ell L_\ell^\alpha(t) - (\ell+\alpha)L_{\ell-1}^\alpha(t), \\
&t\,\mathrm{d}/\mathrm{d}t\, L_\ell^\alpha(t) = (\ell+1)L_{\ell+1}^\alpha(t) - (\ell+\alpha+1-t)L_\ell^\alpha(t),
\end{aligned}$$

we obtain

$$\pi^N_{\mathbf{m},k}(\mathbf{k})\big\{e^{-t/2}L_\ell^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t)\big\} = \big(|\mathbf{m}| + \frac{N}{2} + \gamma_k + 2\ell\big)e^{-t/2}L_\ell^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t), \tag{3.24}$$

$$\pi^N_{\mathbf{m},k}(\mathbf{n}^+)\big\{e^{-t/2}L_\ell^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t)\big\} = (\ell+1)e^{-t/2}L_{\ell+1}^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t), \tag{3.25}$$

$$\pi^N_{\mathbf{m},k}(\mathbf{n}^-)\big\{e^{-t/2}L_\ell^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t)\big\} = -\big(|\mathbf{m}| + \frac{N}{2} + \gamma_k + \ell - 1\big)e^{-t/2}L_{\ell-1}^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t). \tag{3.26}$$

Now the statement holds by means (3.14). $\square$

Next we turn our attention to the exponentiation of the representation ω_k. An operator $\mathcal{O}$ is called essentially self-adjoint, if it is symmetric and its closure is self-adjoint. Let $\mathcal{O}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ with domain $\mathbb{D}(\mathcal{O})$, and let $\{f_n\}_n$ be a complete orthonormal set in $\mathcal{H}$. If each $f_n \in \mathbb{D}(\mathcal{O})$ and there exists $\lambda_n \in \mathbb{R}$ such that $\mathcal{O}f_n = \lambda_n f_n$, for every n, then $\mathcal{O}$ is essentially self-adjoint and the spectrum of its closure $\overline{\mathcal{O}}$, which is a self-adjoint operator, is given by $\operatorname{Spec}(\overline{\mathcal{O}}) = \{\lambda_n \mid n \in \mathbb{Z}\}$. We refer to [6, Chapter 1] for more details on this matter.

We shall also, for ease reference, recall Nelson's result [21]. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, and $\mathfrak{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. A skew-symmetric $\mathfrak{g}$-module ω is said to be integrable, if there exists a continuous unitary representation Ω of $\mathfrak{G}$ in a Hilbert space $\mathcal{H}$ such that $\omega = d\Omega$. Note that if $\omega = d\Omega_1 = d\Omega_2$, then $\Omega_1 = \Omega_2$. The following is a reformulation of a beautiful theorem due to Nelson [21].

Theorem 3.11. *Let $X_1, \dots, X_l$ be a basis of $\mathfrak{g}$ and ω a densely defined $\mathfrak{g}$-module in $\mathcal{H}$. Then $\omega = d\Omega$ for some continuous unitary representation Ω of $\mathfrak{G}$ if and only if* (i) *for all $X \in \mathfrak{g}$, $\omega(X)$ is a skew-symmetric operator on $\mathcal{H}$, and* (ii) *the operator $\omega(X_1^2 + \cdots + X_l^2)$ is essentially self-adjoint.*

Henceforth, $\mathfrak{G}$ denotes the simply connected covering Lie group with Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$.

Theorem 3.12. *The representation ω_k exponentiates to define a unique unitary representation Ω_k of $\mathfrak{G}$ on $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$.*

Proof. Let $\mathbf{u_1} = \mathbf{e}^+ - \mathbf{e}^-$, $\mathbf{u_2} = \mathbf{e}^+ + \mathbf{e}^-$ and $\mathbf{u_3} = \mathbf{h}$, so that $\{\mathbf{u_i}\}_\mathbf{i}$ is a basis of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. In particular

$$-\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2 = \mathbf{h}^2 + 2\mathbf{e}^+\mathbf{e}^- + 2\mathbf{e}^-\mathbf{e}^+ = \mathscr{C}, \qquad \text{and} \qquad \mathbf{u_1}^2 = -\mathbf{k}^2.$$

By Corollary 3.8 and Proposition 3.10, the elements of the orthonormal basis $\{\phi_{\ell,\mathbf{m},j} : \ell \in \mathbb{N}, \mathbf{m} \in \mathbb{Z}_+^N, j \in J_{|\mathbf{m}|}\}$ of $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$ are eigenvectors of

$$\omega_k(\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2) = \omega_k(\mathscr{C} - 2\mathbf{k}^2),$$

and the eigenvalues are real. Thus, the operator $\omega_k(\sum_\mathbf{i} \mathbf{u_i}^2)$ is essentially self-adjoint. Moreover, by Proposition 3.9, $\omega_k(X)$ is skew-symmetric for all $X \in \mathfrak{g}$. It follows then from Theorem 3.11 that ω_k exponentiates to define on $L^2(\mathbb{R}^N, \vartheta_k(x)dx)$ a unique unitary representation Ω_k of the simply connected Lie group $\mathfrak{G}$. $\square$

Remark 3.13. The above theorem proves the claim stated in [3] about the existence of the integrated representation Ω_k (recall the INTRODUCTION).

Remark 3.14. (i) Set $\triangle = \mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2$. Since the elements $\phi_{\ell,\mathbf{m},j}$ are eigenfunctions normalized to one of $\omega_k(\triangle)$, it follows that these eigenfunctions are analytic vectors for $\omega_k(\triangle)$. Thus, by virtue of [21, Theorem 3], the set $\{\phi_{\ell,\mathbf{m},j}\}$ provides a dense set of analytic vectors for the representation Ω_k of $\mathfrak{G}$.

(ii) After this paper was finished, it came to our attention another beautiful theory of integrability of Lie algebras representations elaborated by Flato, Simon, Snellman and Sternheimer [13]. In contrast to Nelson's theory it gives integrability criteria in terms of the properties of the generators of the Lie algebra. Generally, Flato *et al.*'s criteria is more effective in practical applications, especially for higher dimensional Lie algebras.

(iii) As suggested by the referee, one may also use the explicit action of $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ on $\mathcal{S}(\mathbb{R}^+)$, and the euclidean Fourier transform to prove the integrability of ω_k.

Since $\mathfrak{G}$ is simply connected, the following map $\mathbb{R} \longrightarrow K$, $t \mapsto \exp(t(\mathbf{e}^- - \mathbf{e}^+))$, is a diffeomorphism. Moreover, if $\mathscr{Z}$ denotes the center of $\mathfrak{G}$, then $\mathscr{Z} \subset K$, and

$$\mathscr{Z} = \{\exp(r\pi(\mathbf{e}^- - \mathbf{e}^+)) \mid r \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

This is a consequence of the fact that $\mathscr{Z}$ is the kernel of the adjoint representation and the fact that $(\mathbf{e}^- - \mathbf{e}^+) = -i\mathbf{k}$. On the other hand, by Proposition 3.10, for every element $\exp(t(\mathbf{e}^- - \mathbf{e}^+)) = \exp(-it\mathbf{k}) \in K$ we have

$$\Omega_k(\exp(t(\mathbf{e}^- - \mathbf{e}^+)))\phi_{\ell,\mathbf{m},j} = \Omega_k(\exp(-it\mathbf{k}))\phi_{\ell,\mathbf{m},j} = e^{-it(|\mathbf{m}|+\gamma_k+\frac{N}{2}+2\ell)}\phi_{\ell,\mathbf{m},j}.$$

The following two facts then hold:

(i) For $\gamma_k+\frac{N}{2} \in \mathbb{N}$, the element $\exp(r\pi(\mathbf{e}^- - \mathbf{e}^+)) \in \operatorname{Ker}\Omega_k$ if and only if $r\pi(\mathbf{e}^- - \mathbf{e}^+) \in \mathscr{Z}^2$.

(ii) For $\gamma_k+\frac{N}{2} \in \frac{\mathbb{N}}{2}$, the element $\exp(r\pi(\mathbf{e}^- - \mathbf{e}^+)) \in \operatorname{Ker}\Omega_k$ if and only if $r\pi(\mathbf{e}^- - \mathbf{e}^+) \in \mathscr{Z}^4$.

For $d = 1, 2, \ldots$, the quotient $\mathfrak{G}/\mathscr{Z}^d$ is the d-th fold covering of the adjoint group $PSL(2,\mathbb{R})$. We may identify

$$SL(2,\mathbb{R}) \equiv \mathfrak{G}/\mathscr{Z}^2 \quad \text{and} \quad Mp(2,\mathbb{R}) \equiv \mathfrak{G}/\mathscr{Z}^4,$$

where $Mp(2,\mathbb{R})$ is the metaplectic group, i.e. the double covering of $SL(2,\mathbb{R})$. In the light of all the above discussions, the following holds.

Proposition 3.15. *For all $k \in \mathscr{K}^+$ we have:*

(i) *The unitary representation Ω_k descends to $SL(2,\mathbb{R})$ if and only if $\gamma_k + \frac{N}{2} \in \mathbb{N}$.*

(ii) *The unitary representation Ω_k descends to $Mp(2,\mathbb{R})$ if and only if $\gamma_k + \frac{N}{2} \in \frac{\mathbb{N}}{2}$.*

(iii) *The unitary representation Ω_k descends to the universal covering $\widetilde{SL(2,\mathbb{R})}$ if and only if $\gamma_k + \frac{N}{2} \in \mathbb{R}$.*

Recall that the Dunkl transform on the space $L^1(\mathbb{R}^N, \vartheta_k(x)dx)$ is given by

$$\mathscr{D}_k f(\xi) = c_k^{-1}\int_{\mathbb{R}^N} f(x)E_k(x,-i\xi)\vartheta_k(x)dx, \qquad \xi \in \mathbb{R}^N.$$

Next we will prove that the Dunkl transform can be written as

$$\mathscr{D}_k = e^{i\frac{\pi}{2}(\gamma_k+\frac{N}{2})}e^{-i\frac{\pi}{4}(\|x\|^2-\Delta_k)}. \tag{3.27}$$

That is, up to a constant, $\mathscr{D}_k$ is an element of the integrated form of the representation ω_k, formulated above. This claim was proved earlier in [3, Corollary 4.14] using a generalized Segal-Bargmann transform associated with the finite reflection group G. However, for simplicity and completeness, we shall present here another argument.

As an immediate consequence of Proposition 3.10, we have

$$\Omega_k\big(\exp(-i\frac{\pi}{2}\mathbf{k})\big)\phi_{\ell,\mathbf{m},j}(x) = (-i)^{2\ell+|\mathbf{m}|}e^{-i\frac{\pi}{2}(\gamma_k+\frac{N}{2})}\phi_{\ell,\mathbf{m},j}(x). \tag{3.28}$$

On the other hand, by [11, Theorem 2.6], the Dunkl transform of $\phi_{\ell,\mathbf{m},j}$ is given by

$$\mathscr{D}_k(\phi_{\ell,\mathbf{m},j})(x) = (-i)^{2\ell+|\mathbf{m}|}\phi_{\ell,\mathbf{m},j}(x). \tag{3.29}$$

Now, since $e^{i\frac{\pi}{2}(\gamma_k+\frac{N}{2})}\Omega_k\big(\exp(-i\frac{\pi}{2}\mathbf{k})\big)$ and $\mathscr{D}_k$ are unitary operators on $L^2(\mathbb{R}^N, \vartheta(x)dx)$, (3.27) follows from (3.28) and (3.29).

Remark 3.16. From a representation theory point of view, representing a Fourier-type transform by a group element, up to a constant, it is not a surprising phenomena. See for instance [8, 15].

In the light of (3.27), we may define a transform $\mathscr{F}^N_{\mathbf{m},k}$ on $\mathcal{S}(\mathbb{R}^+)$ by

$$\begin{aligned} \alpha^N_{\mathbf{m},k}\big(h\otimes \mathscr{F}^N_{\mathbf{m},k}(\psi)\big) &:= \alpha^N_{\mathbf{m},k}\Big(h\otimes \Pi^N_{\mathbf{m},k}\big(e^{i\frac{\pi}{2}(\gamma_k+N/2)}\exp(-i\frac{\pi}{2}\mathbf{k})\big)\psi\Big) \\ &= \mathscr{D}_k\big(\alpha^N_{\mathbf{m},k}(h\otimes\psi)\big), \qquad \text{(by (3.14) and (3.27))} \end{aligned}$$

where $h\in\mathcal{H}_{|\mathbf{m}|,k}$, and $\Pi^N_{\mathbf{m},k}$ is the unique unitary representation of the Lie group $\mathfrak{G}$ such that $d\Pi^N_{\mathbf{m},k}=\pi^N_{\mathbf{m},k}$ (see Theorem 3.19 bellow for more details). Now the following holds.

Theorem 3.17. (Bochner-type formula) *Let $k\in\mathscr{K}^+$. Then we have:*

(i) *If $f(x)=h(x)\psi(\|x\|^2)$, with $h\in\mathcal{H}_{|\mathbf{m}|,k}$ and $\psi\in\mathcal{S}(\mathbb{R}^+)$, then*

$$\mathscr{D}_k(f)(\xi)=h(\xi)\mathscr{F}^N_{\mathbf{m},k}(\psi)(\|\xi\|^2),$$

where $\mathscr{F}^N_{\mathbf{m},k}$ depends only on $|\mathbf{m}|+\frac{N}{2}+\gamma_k$, up to a constant, i.e.

$$e^{-i\frac{\pi}{2}(\gamma_k+\frac{N}{2})}\mathscr{F}^N_{\mathbf{m},k}=e^{-i\frac{\pi}{2}(\gamma_{k'}+\frac{N'}{2})}\mathscr{F}^{N'}_{\mathbf{m}',k'}$$

if

$$|\mathbf{m}|+\frac{N}{2}+\gamma_k=|\mathbf{m}'|+\frac{N'}{2}+\gamma_{k'}. \tag{3.30}$$

(ii) *The transform $\mathscr{F}^N_{\mathbf{m},k}$ coincides with the classical Hankel transform. More precisely, for $\psi\in\mathcal{S}(\mathbb{R}^+)$,*

$$\mathscr{F}^N_{\mathbf{m},k}(\psi)(r^2)=e^{-i\frac{\pi}{2}|\mathbf{m}|}\mathscr{H}_{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(\psi\circ\Upsilon)(r), \tag{3.31}$$

where $\Upsilon(t):=t^2$ for $t\in\mathbb{R}$, and

$$\mathscr{H}_\alpha f(r):=\int_0^\infty f(s)\frac{J_\alpha(rs)}{(rs)^\alpha}s^{2\alpha+1}ds$$

denotes the Hankel transform, with J_α is the Bessel function of the first kind. In these circumstances, the statement (i) reads

$$\mathscr{D}_k(h\psi(\|\cdot\|))(\xi)=e^{-i\frac{\pi}{2}|\mathbf{m}|}h(\xi)\mathscr{H}_{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1}(\psi)(\|\xi\|),$$

for $h\in\mathcal{H}_{|\mathbf{m}|,k}$ and $\psi\in\mathcal{S}(\mathbb{R}^+)$.

Proof. The first statement holds from the definition of $\mathscr{F}^N_{\mathbf{m},k}$, and from Lemma 3.5. To prove (ii), let us start with $\mathbf{m}\in\mathbb{Z}^N_+$ such that $|\mathbf{m}|=0$, i.e. $\mathbf{m}=\mathbf{0}$. In this case $f(x)=\psi(\|x\|^2)$, i.e. f is a radial function. This case was basically done in [28] (see also [24]). However, for completeness we shall briefly include the argument for radial functions. Assume that $F(x)=F_\circ(\|x\|)$. Using the polar coordinates and the homogeneity of ϑ_k, we have

$$\mathscr{D}_k(F)(\xi)=c_k^{-1}\int_0^\infty F_\circ(r)r^{2\gamma_k+N-1}\Big\{\int_{\mathbb{S}^{N-1}}E_k(-ir\xi,\theta)\vartheta_k(\theta)d\omega(\theta)\Big\}dr.$$

By [28, Corollary 2.2], we have

$$\int_{\mathbb{S}^{N-1}}E_k(-ir\xi,\theta)\vartheta_k(\theta)d\omega(\theta)=c_k\frac{J_{\gamma_k+N/2-1}(r\|\xi\|)}{(r\|\xi\|)^{\gamma_k+N/2-1}}.$$

Thus $\mathscr{D}_k(F)(\xi) = \mathscr{H}_{\gamma_k+N/2-1}(F_\circ)(\|\xi\|)$. This implies that

$$\mathscr{D}_k(f)(\xi) = \mathscr{H}_{\gamma_k+\frac{N}{2}-1}(\psi \circ \Upsilon)(\|\xi\|)$$

whenever $f(x) = \psi(\|x\|^2)$, and therefore $\mathscr{F}^N_{\mathbf{0},k}(\psi)(\|\xi\|^2) = \mathscr{H}_{\gamma_k+\frac{N}{2}-1}(\psi \circ \Upsilon)(\|\xi\|)$. Now let $\mathbf{m} \in \mathbb{Z}^N_+$ such that $|\mathbf{m}| \neq 0$. Equation (3.30) gives

$$\mathscr{F}^N_{\mathbf{m},k}(\psi)(r^2) = e^{-i\frac{\pi}{2}|\mathbf{m}|}\mathscr{F}^{2|\mathbf{m}|+N}_{\mathbf{0},k}(\psi)(r^2) = e^{-i\frac{\pi}{2}|\mathbf{m}|}\mathscr{H}_{\gamma_k+\frac{N}{2}+|\mathbf{m}|-1}(\psi \circ \Upsilon)(r).$$

□

Example 3.18. (Hecke-type formula) If $\psi(s) = e^{-\frac{s^2}{2}}$, then

$$\begin{aligned}\mathscr{D}_k\big(e^{-\frac{\|x\|^2}{2}}h\big)(\xi) &= e^{-i\frac{\pi}{2}|\mathbf{m}|}h(\xi)\mathscr{H}_{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1}\big(e^{-\frac{s^2}{2}}\big)(\|\xi\|) \\ &= e^{-i\frac{\pi}{2}|\mathbf{m}|}e^{-\frac{\|\xi\|^2}{2}}h(\xi).\end{aligned}$$

Thus, we recover the Hecke formula for the Dunkl transform which was initially proved by Dunkl in [11], and later in [3] by Ørsted and the present author using an $\mathfrak{sl}(2,\mathbb{R})$-argument similar to the one illustrated above.

We shall give two representation formulas for $\mathscr{F}^N_{\mathbf{m},k}$, when $2\gamma_k + N \in \mathbb{N}$. From (3.31), substituting $t = r^2$, we have

$$\mathscr{F}^N_{\mathbf{m},k}(\psi)(t) = e^{-i\frac{\pi}{2}|\mathbf{m}|}\int_0^\infty \psi(s^2)\frac{J_{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1}(s\sqrt{t})}{(\sqrt{t})^{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1}}s^{|\mathbf{m}|+\gamma_k+\frac{N}{2}}ds.$$

Using the following well known formula

$$\left(\frac{d}{dz}\right)^{\aleph}\big[z^{-(\nu-1)}J_{\nu-1}(z)\big] = (-1)^{\aleph}z^{-(\nu-1)}J_{\nu-1+\aleph}(z),$$

we obtain

$$\frac{J_\nu(\sqrt{t}s)}{(\sqrt{t})^\nu} = \left(-\frac{2}{s}\right)^{\aleph}\frac{d^{\aleph}}{dt^{\aleph}}\left\{\frac{J_{\nu-\aleph}(\sqrt{t}s)}{(\sqrt{t})^{\nu-\aleph}}\right\}.$$

Hence, the bellow integral representation for $\mathscr{F}^N_{\mathbf{m},k}$ holds

$$\mathscr{F}^N_{\mathbf{m},k}(\psi)(t) = e^{-i\frac{\pi}{2}|\mathbf{m}|}(-2)^{\aleph}\frac{d^{\aleph}}{dt^{\aleph}}\left\{\int_0^\infty \psi(s^2)\frac{J_{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1-\aleph}(\sqrt{t}s)}{(\sqrt{t})^{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1-\aleph}}s^{|\mathbf{m}|+\gamma_k+\frac{N}{2}-\aleph}ds\right\}.$$

This expression leads to the following two representation formulas for the transform $\mathscr{F}^N_{\mathbf{0},k}$ (and thus for $\mathscr{F}^N_{\mathbf{m},k}$) when $2\gamma_k + N \in \mathbb{N}$:

(i) If $2\gamma_k + N = 2\aleph + 1$,

$$\mathscr{F}^N_{\mathbf{0},k}(\psi)(t) = \left(\frac{2}{\pi}\right)^{1/2}(-2)^{\aleph}\frac{d^{\aleph}}{dt^{\aleph}}\left\{\int_0^\infty \psi(s^2)\cos(\sqrt{t}s)ds\right\}.$$

(ii) If $2\gamma_k + N = 2\aleph + 2$,

$$\mathscr{F}^N_{\mathbf{0},k}(\psi)(t) = (-2)^{\aleph}\frac{d^{\aleph}}{dt^{\aleph}}\left\{\int_0^\infty \psi(s^2)J_0(\sqrt{t}s)sds\right\}.$$

Question. In [4], the authors proved that the wave equation associated with Δ_k satisfies the strict Huygens' principle if and only if $\gamma_k + (N-3)/2 \in \mathbb{N}$. Does there exist a link between the validity of Huygens' principle and the fact that if $2\gamma_k + N$ is odd then $\mathscr{F}^N_{\mathbf{0},k}$ is given in terms of the cosine transform, while if $2\gamma_k + N$ is even then $\mathscr{F}^N_{\mathbf{0},k}$ is given in terms of the classical Hankel transform?

We conclude this paper by putting together a few facts regarding the representation $\pi^N_{\mathbf{m},k}$, since they are immediate consequences of the results previously elaborated.

For $s \in \mathbb{R}$, write $\mathtt{e}_s = \exp_{\mathfrak{G}}(s\mathbf{e}^+)$, $\mathtt{h}_s = \exp_{\mathfrak{G}}(s\mathbf{h})$, and $\varkappa = \exp_{\mathfrak{G}}(\frac{\pi}{2}(\mathbf{e}^- - \mathbf{e}^+)) = \exp_{\mathfrak{G}}(-i\frac{\pi}{2}\mathbf{k})$. Here $\exp_{\mathfrak{G}}$ denotes the exponential map of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{G}$.

Recall that the Laguerre polynomials satisfy the orthogonality relation

$$\int_0^\infty e^{-t} L^\alpha_\imath(t) L^\alpha_\jmath(t) t^\alpha dt = \delta_{\imath\jmath} \frac{\Gamma(\alpha+\imath+1)}{\Gamma(\imath+1)}.$$

Theorem 3.19. *Let $k \in \mathscr{K}^+$ and $\mathbf{m} \in \mathbb{Z}^N_+$.*

(i) *The dense subspace, in $L^2(\mathbb{R}^+, t^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}dt)$, spanned by the Laguerre functions $\big\{e^{-t/2} L_\ell^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}(t)\big\}_{\ell\in\mathbb{N}}$, is stable under the action of $\pi^N_{\mathbf{m},k}(\mathfrak{sl}(2,\mathbb{C}))$, and the spectrum of $\pi^N_{\mathbf{m},k}(\mathbf{k})$ is positive.*

(ii) *There exists a unique unitary representation $\Pi^N_{\mathbf{m},k}$ of the simply connected Lie group $\mathfrak{G}$ on $L^2(\mathbb{R}^+, t^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1}dt)$, such that $\pi^N_{\mathbf{m},k}(X)f = (\mathrm{d}/\mathrm{d}s)_{|s=0}\Pi^N_{\mathbf{m},k}(\exp(sX))f$ for $f \in \mathcal{S}(\mathbb{R}^+)$. Further, the unitary representation $\Pi^N_{\mathbf{m},k}$ may be described by the formulas:*

① $\Pi^N_{\mathbf{m},k}(\mathtt{e}_s)f : t \mapsto e^{its/2} f(t)$;

② $\Pi^N_{\mathbf{m},k}(\mathtt{h}_s)f : t \mapsto e^{(|\mathbf{m}|+\gamma_k+N/2)s} f(e^{2s}t)$;

③ $\Pi^N_{\mathbf{m},k}(\varkappa)f : t \mapsto \dfrac{1}{2} e^{-i\frac{\pi}{2}(|\mathbf{m}|+\gamma_k+N/2)} \bar{\mathscr{H}}_{|\mathbf{m}|+\gamma_k+\frac{N}{2}-1}(f)(t)$, *where $\bar{\mathscr{H}}_\alpha$ is the Hankel-type transform given by*

$$\bar{\mathscr{H}}_\alpha f(t) = \int_0^\infty f(u) \frac{J_\alpha((ut)^{1/2})}{(ut)^{\alpha/2}} u^\alpha du,$$

where J_α is the Bessel function of the first kind.

Proof. (i) This statement is just (3.24), (3.25), and (3.26).
(ii) The existence and the uniqueness of $\Pi^N_{\mathbf{m},k}$ follow from Theorem 3.12 and (3.14). However, in the light of the statement (i), one may also prove the integrability of $\pi^N_{\mathbf{m},k}$ in a direct fashion using Nelson's theorem, as we did previously with ω_k. Both formulas ① and ② are clear. Formula ③ follows from the definition of $\mathscr{F}^N_{\mathbf{m},k}$ and (3.31). $\square$

Remark 3.20. Let $\varrho_0(t) := e^{-t/2} L_0^{|\mathbf{m}|+N/2+\gamma_k-1}(t) = e^{-t/2}$ be the vector annihilated by $\pi^N_{\mathbf{m},k}(\mathbf{n}^-)$ (see (3.26)). A simple change a variable gives

$$\begin{aligned} \big(\Pi^N_{\mathbf{m},k}(\mathtt{h}_s)\varrho_0, \varrho_0\big) &:= \int_{\mathbb{R}^+} \big(\Pi^N_{\mathbf{m},k}(\mathtt{h}_s)\varrho_0(t)\big)\, \varrho_0(t) t^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1} dt \\ &= \int_{\mathbb{R}^+} e^{-t\cosh(s)} t^{|\mathbf{m}|+\frac{N}{2}+\gamma_k-1} dt \\ &= \Gamma(|\mathbf{m}| + \frac{N}{2} + \gamma_k) \cosh(s)^{-(|\mathbf{m}|+\frac{N}{2}+\gamma_k)}. \end{aligned}$$

On the other hand, every element of $\mathfrak{G}$ can be written as $\exp(-i\theta_1\mathbf{k})\mathfrak{h}_s\exp(-i\theta_2\mathbf{k})$, with $\theta_1,\ \theta_2 \in \mathbb{R}$ and $s \geq 0$, and the corresponding Haar measure is $|e^{2s} - e^{-2s}|dsd\theta_1 d\theta_2$ (cf. [1, 25]). This decomposition of $\mathfrak{G}$ and formula (3.24) imply that $\Pi^N_{\mathbf{m},k}$ is square integrable on $\mathfrak{G}/\mathcal{Z}$ only if $|\mathbf{m}| + N/2 + \gamma_k > 1$.

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